Regulating synchronous states of complex networks by pinning interaction with an external node

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To shed light on how biological and technological systems can establish or maintain a synchronous functioning, we address the problem of how to engineer an external pinning action on a network of dynamical units. In particular, we study the regulation of a network toward a synchronized behavior by means of a bidirectional interaction with an external node that leaves unchanged its inner parameters and architecture. We demonstrate that there are two classes of networks susceptible of being regulated into a synchronous motion and provide a simple method, for each one of them, to properly design a pinning sequence to achieve regulation. We also discuss how the obtained sequence can be compared with a topological ranking of the network nodes.

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Synchronous behavior of networking dynamical units [1] and its regulation is being the focus of intense research [2], especially in technology and biology, where it is nowadays considered an essential process through which such networks establish and maintain their correct functioning. It is, for instance, the case of distributed computation, where causality and reproducibility of massive parallel discrete-time event simulations need the architecture of the communication links to ensure synchronization in the absence of a global pacemaker [3]. On the other side, in wireless communication networks, synchronized time activity is a must for purposes such as location, proximity and energy efficiency in information routing or environmental sensing [4]; in electrical power distribution lines of generators and consumers, desynchronization should be prevented to avoid massive power blackouts [5]. In biology, understanding the regulatory processes underlying the synchronous behavior can offer insights on how these systems [6] enhance and/or maintain their synchronous functionality.

In this paper, we address the problem of how to properly engineer an external pinning action on a network, constrained to leave unchanged its dynamical parameters and topology. Specifically, we consider a network of identical dynamical units, whose synchronous behavior is regulated by a *pinning interaction* with an external node, i.e., by the addition of a further identical system forming as few as possible *bidirectional* connections with nodes in the original graph. In particular, we will focus on sequential regulation processes in order to establish conditions for identifying the minimal number of connections needed for regulating synchrony, as well as a practical way to find the corresponding sequence of connections. We will furthermore discuss how the regulation process is related to some of the topological ranking of the nodes in the original graph.

The problem under study can be conveniently described by introducing an initial graph $\mathcal{G}_0$ of N bidirectionally coupled identical systems, represented by a vector state $\mathbf{x}_i(t) \in \mathbb{R}^m$ $(i=1,\ldots,N)$ whose evolution equation is

$$\dot{\mathbf{x}}_i = f(\mathbf{x}_i) + \sigma \sum_{j=1}^{N} \mathcal{L}_{ij} h(\mathbf{x}_j),$$

in which dot denotes temporal derivative, $\mathcal{L}=(\mathcal{L}_{ij}) \in \mathcal{M}_N$ is the Laplacian matrix associated with the network $\mathcal{G}_0$, $f(\mathbf{x}):\mathbb{R}^m \to \mathbb{R}^m$ is the function ruling the local evolution, $h(\mathbf{x}):\mathbb{R}^m \to \mathbb{R}^m$ is the output function of the node, and $\sigma \in \mathbb{R}^+$ is the strength of the bidirectional coupling, which, from now on, will be such that the synchronous state $\mathbf{x}_s(t) = \mathbf{x}_1(t) = \ldots = \mathbf{x}_N(t)$ is linearly unstable.

In order to regulate the stability of $\mathbf{x}_s(t)$, we consider here an interaction between $\mathcal{G}_0$ and a *unique* external dynamical system, identical to those in $\mathcal{G}_0$, that forms at successive times $t_n(n=1,\ldots,N)$ a series of σ-strength *bidirectional* connections by pinning the nodes in $\mathcal{G}_0$ with a given sequence $\{s_1,s_2,\ldots,s_N\}$. The new equation of motion then reads as

$$\dot{\mathbf{x}}_i = f(\mathbf{x}_i) + \sigma \sum_{j=1}^{N+1} \mathcal{L}'_{ij}(t) h(\mathbf{x}_j), \tag{1}$$

where $\mathcal{L}'(t)=(\mathcal{L}'_{ij}(t)) \in \mathcal{M}_{N+1}$ is now the following time dependent Laplacian matrix

$$\begin{pmatrix} \mathcal{L}'_{11}(t) & \mathcal{L}'_{12} & \cdots & \mathcal{L}'_{1N} & \Theta(t-T_1) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathcal{L}'_{N1} & \mathcal{L}'_{N2} & \cdots & \mathcal{L}'_{NN}(t) & \Theta(t-T_N) \\ \Theta(t-T_1) & \cdots & \cdots & \Theta(t-T_N) & \mathcal{L}'_{N+1,N+1}(t) \end{pmatrix}$$

whose elements are such that

(i) $\mathcal{L}'_{ij}=\mathcal{L}_{ij}$ for $i \neq j$ and $i,j < N+1$;

(ii) $\Theta(t-T_i)=\mathcal{L}'_{i,N+1}(t)=\mathcal{L}'_{N+1,i}(t)$, being T_i the time at which the i^{th} node in $\mathcal{G}_0$ is pinned by the interaction with the external node, and Θ the Heaviside function. Notice that while the index i in t_i refers to a time ordering, the index i in T_i points to the ordering of the pinning sequence; thus, $t_i = T_{s_i}$.

(iii) $\mathcal{L}'_{ii}(t) = -\sum_{j\neq i} \mathcal{L}'_{ij}$.

The key point is the way in which the sequence $\{s_1,s_2,\ldots,s_N\}$ of links is established to ensure regulation of the synchronous behavior of $\mathcal{G}_0$. Since the coupling matrix $\mathcal{L}'(t)$ is a symmetric, zero row-sum matrix at all times, a natural starting point is the master stability function (MSF) approach [7] that allows us to assess the new stability property of the invariant synchronous manifold $\mathbf{x}_1(t)=\ldots=\mathbf{x}_N(t)=\mathbf{x}_{N+1}(t)=\mathbf{x}_s(t)$ of Eq. (1). In particular, Ref. [8] fully discusses the application of the MSF formalism to the assessment of the so-called *network synchronizability* (i.e., the ability of the network to sustain a stable synchronous behavior) and demonstrates that only two possible classes of dynamical systems allow stability of the synchronous behavior: the so-called *class II* systems, where the stability of the synchronous state is guaranteed above a given threshold in the coupling strength σ, and *class III* systems, where the stability of the synchronous state is obtained within a given interval of coupling strength values [9]. In the former case, the role of the network's architecture is that of setting a threshold $\sigma_c \equiv \frac{\mu_1}{\lambda_2}$, with μ_1 being the value of the MSF parameter at which the MSF itself passes from positive to negative values of the conditional maximum Lyapunov exponent [8], and λ_2 is the second smallest eigenvalue of the network's coupling matrix. In the latter case, class III systems, the synchronous network behavior is stable between μ_1 and μ_2, so that its network synchronizability is fully accounted for by the ratio between the largest and the second smallest eigenvalue in the spectrum of the network's coupling matrix, $\frac{\lambda_{N+1}}{\lambda_2}$.

However, as regulation of synchronization is concerned, these two different classes of functions need to be treated separately. We will show that, in general, network synchronizability leads to wrong conclusions since one of the constraints in regulation is to keep the coupling strength fixed. To this purpose we will accompany, from now on, our analysis with numerical examples, corresponding to the case of $N=400$ nodes arranged in two different network configurations: a small-world (SW) network obtained as in Ref. [10] by initially arranging the N nodes in a ring with connections only between nearest neighbors, and by randomly adding with probability $p=0.02$ a connection between unconnected pairs of nodes (thus with an average degree $\langle k\rangle=2+pN=10$), and a scale-free (SF) network obtained by the preferential attachment process of Ref. [11] with the same average degree. Furthermore, we will consider networks of coupled Rössler systems [12], $f(\mathbf{x}\equiv(x,y,z))=[-y-z,x+0.165y,0.2+z(x-10)]$, in Eq. (1) to allow a direct comparison of class III and class II systems by selecting the appropriate output function. Actually, when $h[\mathbf{x}\equiv(x,y,z)]=[x,0,0]$ a class III system is obtained with $\mu_1=0.206$ and $\mu_2=5.519$, while $h[\mathbf{x}\equiv(x,y,z)]=[0,y,0]$ determines a class II system with $\mu_1=0.178$.

Let us proceed first with the analysis of the class II system. The initial configuration is a unstable synchronous motion in which $\lambda_2^{ini}<\frac{\mu_1}{\sigma}$, being λ_2^{ini} the second smallest eigenvalue of $\mathcal{L}$ [i.e., λ_3 of $\mathcal{L}'(0)$]. Since the larger is λ_2, the smaller the threshold for setting a stable synchrony, a practical way to regulate the synchronous state is as follows: at time t_1 (at which the first connection is established between the regulating node and $\mathcal{G}_0$) we consider all possible choices and fix s_1 as the index of the node for which one obtains the maximum value for the second smallest eigenvalue $\lambda_2(1)$ in the matrix $\mathcal{L}'(T_{s_1})$. After establishing this connection, at time t_2 one looks for that index $s_2\neq s_1$ for which again the matrix $\mathcal{L}'(T_{s_2})$ displays a maximum value of $\lambda_2(2)$. At time t_3 the process is repeated to select $s_3\neq s_1,s_2$, and so on, until the full sequence $\{s_1,s_2,\ldots,s_N\}$ is determined.

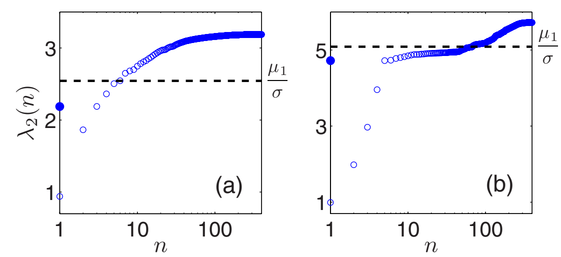

FIG. 1. (Color online) Log-linear graph showing the behavior of $\lambda_2(n)$ during the regulation of a class II system for (a) SW and (b) SF networks (see text for details). As reference, the values of λ_2^{ini} (full circles) and ratios μ_1/σ (dashed lines) with (a) $\sigma=0.07$ and (b) $\sigma=0.045$ are given.

Figure 1 reports the values of the second smallest eigenvalue $\lambda_2(n)$ *vs.* the number n of links formed by the regulating node with nodes in $\mathcal{G}_0$, for the above described procedure applied to the SW [panel (a)] and SF [panel (b)] networks. It shows that the method allows, indeed, for the identification in both cases of a locally minimum number (and the corresponding minimal sequence) of links needed to regulate the synchronous state. A synoptic view of the two panels in Fig. 1 reveals that, in class II, SW networks are *easier* to regulate with respect to SF ones in that the corresponding number of links to make the synchronous state stable is smaller, and that synchronizability is monotonically enhanced with n as the behavior of the synchronization threshold $\frac{\mu_1}{\lambda_2(n)}$ is a decreasing function of n. These results, therefore, are fully consistent with what has been recently reported in a degree driven pinning control process of class II networks [13], for which the existence of a threshold in the number of nodes needed for control was observed.

The situation is completely different in the case of a class III system. Let us start from an unstable synchronous motion in which $\lambda_2^{ini}<\frac{\mu_1}{\sigma}$ and $\lambda_N^{ini}<\frac{\mu_2}{\sigma}$ [14], being λ_N^{ini} the largest eigenvalue of $\mathcal{L}$ [i.e., λ_{N+1} of $\mathcal{L}'(0)$]. Here, the synchronization stability is ensured for all coupling architectures whose corresponding eigenvalue spectrum is entirely contained within the stability region, delimited by the threshold parameters μ_1 and μ_2. Then, the way we select the pinning sequence is that maximizing, at each time t_n, $\lambda_2(n)-\lambda_2^{ini}$ and minimizing $\lambda_{N+1}(n)-\lambda_N^{ini}$ [14], which can be simultaneously achieved by maximizing the following quantity:

$$R(n)=\frac{\lambda_2(n)-\lambda_2^{ini}}{\lambda_{N+1}(n)-\lambda_N^{ini}}. \tag{2}$$

In practice, at time t_1, out of all possible choices, s_1 is the index of the node maximizing $R(1)$. After having established this connection, at time t_2 one looks for that $s_2\neq s_1$ for which the spectrum of $\mathcal{L}'(T_{s_2})$ displays a maximum of $R(2)$, and so on, up to when the regulating sequence $\{s_1,s_2,\ldots,s_N\}$ is determined.

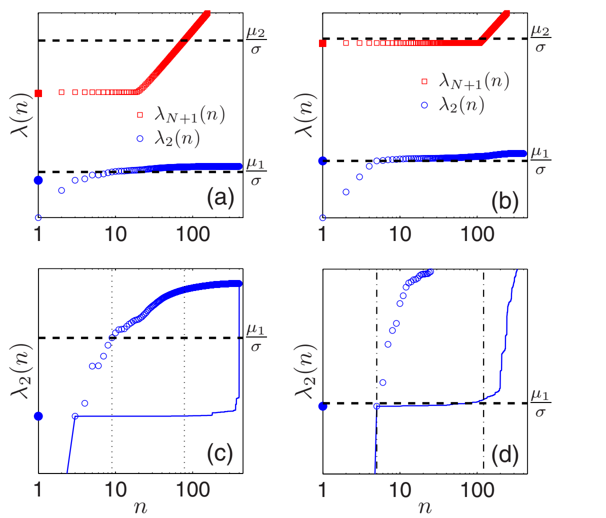

FIG. 2. (Color online) (a) and (b) Log-log behavior of $\lambda_2(n)$ (blue open circles) and $\lambda_{N+1}(n)$ (red open squares) during regulation of a (a) SW network [as in Fig. 1(a)] and (b) SF network [as in Fig. 1(b)] of class III systems. The values of λ_2^{ini} (blue full circle) and λ_N^{ini} (red full square) refer to those of $\mathcal{G}_0$, and the horizontal dashed lines are in correspondence with the limiting thresholds μ_1/σ and μ_2/σ with (a) σ=0.07 and (b) σ=0.045. Panels (c) and (d) are zooms of the $\lambda_2(n)$ behavior shown in (a) and (b), respectively, with vertical (c) dotted and (d) dash-dotted lines marking the initial and final values of n for which the (c) SW and (d) SF networks are regulated. For comparison, $\lambda_2^c(n)$ is also reported in blue solid line as obtained from a ranking centrality-based pinning regulation.

A full account of the regulating process for class III systems is presented in Fig. 2. Namely, panel (a) [(b)] report the behavior of $\lambda_2(n)$ and $\lambda_{N+1}(n)$ at all stages of the selection of the regulating sequence for the chosen SW (SF) network. For both topologies, and at variance with the class II case, class III systems are characterized by the fact that the regulation is only possible up to a maximal regulating sequence. The maximum sequence length is here determined by the monotonic growth of $\lambda_{N+1}(n)$, which eventually gives rise to the emergence of the second instability for synchronization. This is shown in panel (c) [(d)], which is the corresponding enlargement of panel (a) [(b)], by means of vertical dotted (dash-dotted) lines, delimiting the range in which regulation is effective. Notice that SF topologies, being more difficult to regulate, allow on the contrary for a larger number of sequences because the largest eigenvalue crosses later the upper threshold.

The blue solid lines in panels (c) and (d) allow for a direct comparison of the regulating capability of the sequences obtained with Eq. (2), with $\lambda_2^c(n)$ corresponding to the sequence obtained by pinning the nodes of $\mathcal{G}_0$ following a ranking based on the *eigenvector centrality* [15]. Then, and considering that $\lambda_{N+1}^c(n)$ (not shown) behaves very closely to $\lambda_{N+1}(n)$, our sequencing leads, *in all cases*, to a more effective regulating mechanism, indicating that our approach can be used as a way for ranking nodes in a network depending on their importance in regulating the system's collective functioning. Thus, this method allows gathering information on network's nodes that is not contained in other topologically based ranking methods.

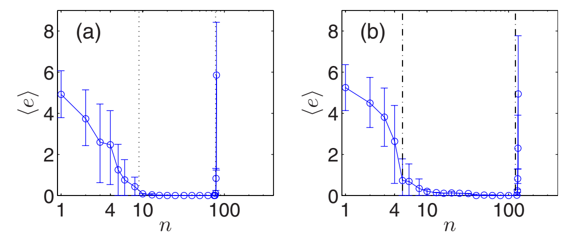

FIG. 3. (Color online) Log-linear plots of the averaged synchronization error $\langle e \rangle$ (see text for definition) *vs.* n during the regulation of the same (a) SW and (b) SF networks used in Fig. 2 following the pinning sequence optimizing $R(n)$ given by Eq. (2). Vertical (a) dotted and (b) dash-dotted lines are added to show the agreement with the predicted range of regulability depicted in Figs. 2(c) and 2(d) respectively.

In order to verify the theoretical findings presented here, we monitor during the regulating process the vanishing of the time average synchronization error

$$e = \frac{1}{TN}\sum_{j=1,\ldots,N}\int_t^{t+T}\|\mathbf{x}_j - \mathbf{x}_{N+1}\|_2 dt$$

over a window T=120 t.u. In Fig. 3 we plot the averaged error $\langle e \rangle$ for 200 different initial conditions of the networking systems (together with their standard deviation as error bars) to show how accurately the predicted regulating ranges are numerically verified.

The existence of a minimal and a maximal regulating sequence for the class III case, the first one due to λ_2 entering the stability region and the second because λ_{N+1} leaves it, implies that an optimal sequence can be found in which there is a balance between these two events. The simplest (but not necessarily optimal) approach to find the regulating sequence is to search for a maximum in $R(n)$, at which the entire spectrum of the network Laplacian better fits within the synchronizability range. This behavior is reported in Fig. 4(a) for a SW (diamonds) and SF (crosses) networks together with the vertical dotted and dash-dotted lines delimiting the respective ranges of regulability.

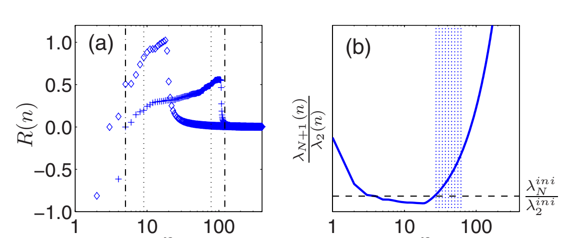

FIG. 4. (Color online) (a) Log-linear behavior of $R(n)$, given by Eq. (2), for the SW (diamonds) and SF (crosses) networks used in Figs. 2(a) and 2(b). The vertical dotted (dash-dotted) lines denote the range of n in which the SW (SF) network is synchronized due to regulation. (b) Log-linear behavior of the ratio $\frac{\lambda_{N+1}(n)}{\lambda_2(n)}$, where $\lambda_2(n)$ and $\lambda_{N+1}(n)$ are those obtained from the regulation process of the SW network of class III systems reported in Fig. 2(a). The shaded region corresponds to regulating sequences whose eigenratio values are larger than $\lambda_N^{ini}/\lambda_2^{ini}$ (horizontal dashed line) at which $\mathcal{G}_0$ displays an asynchronous motion.

The former result differs from previous approaches of *pinning control* of networks (i.e., the situation in which the external node is *unidirectionally* forcing the dynamics of the original graph). There, it was argued that the controllability of a generic network behavior toward an assigned synchronous evolution could be enhanced by pinning configurations that minimize the ratio $\frac{\lambda_{N+1}}{\lambda_2}$ associated to the extended network topology [16] (i.e., increasing the network synchronizability). But the concept of *regulation of synchrony* (i.e., the situation in which the external node is *bidirectionally* coupled to the original graph) described here is fully different from that of *network controllability* since for class III systems many pinning sequences exist that are able to regulate the synchronous behavior of $\mathcal{G}_0$ but correspond to values of $\frac{\lambda_{N+1}}{\lambda_2}$ larger than the initial (unpinned) ratio, for which the synchronous behavior is unstable. To show this, we plot in Fig. 4(b) the behavior of the ratio $\frac{\lambda_{N+1}(n)}{\lambda_2(n)}$, where $\lambda_2(n)$ and $\lambda_{N+1}(n)$ are those obtained from the regulation process of the SW network of class III systems reported in Fig. 2(a). It is clear, from the nonmonotonic trend, that a huge number of still regulating sequences [those within the shaded region in Fig. 4(b)] correspond to eigenratio values that are even *larger* than $\lambda_N^{ini}/\lambda_2^{ini}$ (at which $\mathcal{G}_0$ displayed an asynchronous motion).

The regulating mechanism here presented is not limited to processes where regulation is attained by interaction with an external node, but they can also be applied in all cases in which the problem is to regulate synchrony by minimally rewiring the connections of a given node in the graph. One can, indeed, imagine to start with N networking systems, remove all connections of the node, and substitute them with new connections to the other $N-1$ units of the graph, following the ranking sequence that our criteria are providing.

Finally, although the regulating sequences described in the text were calculated through a steepest descent procedure, which does not guarantee a *globally* minimal solution, we have proved (Fig. 2) that it is enough to derive better sequences than other methods based on topological criteria. Nevertheless, our proposal can also be applied to compute a globally minimal solution since we provide adequate measures to be used as fitness functions for any of the many existing methods in the literature that can solve this optimization problem.

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